Multinomial Pascal's Triangle

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Abstract

We aim to arrange the terms of multinomial expansion in in $\mathbb{R}^n$. This can be seen as a generalization to the well-known Pascal's Triangle. We show that with some constraints derived from observations, there exists a way to arrange terms of multinomial terms in $\mathbb{R}^n$. We further show that there exists a bijection between sets of orthogonal matrices and the set of Multinomial Pascal's Triangle. Furthermore, the orbit of orthogonal matrix on Multinomial Pascal's Triangle generates the entire set of Multinomial Pascal's Triangle

1 Introduction

$$
\begin{aligned}
(a+b)^0 &= \mathbf{1} \\
(a+b)^1 &= \mathbf{1}a + \mathbf{1}b \\
(a+b)^2 &= \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2 \\
(a+b)^3 &= \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3 \\
(a+b)^4 &= \mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4
\end{aligned}
$$

$$
\begin{matrix}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{matrix}
$$

Pascal's Triangle is a smart arrangement of terms of binomial expansions. Each nth row of the triangle contains the coefficients of the binomial expansion of $(a+b)^k$. Normally, the variables a, b are redacted for simplicity. The triangle has two basic interesting properties.

1. The arrangement is visually appealing
2. The value of the binomial coefficient is the sum of the two binomial coefficients directly above it.

I wondered if a similar representation can be made for coefficients of multinomial expansions. The properties above, specifically "visually appealing", and "above" is not really well defined. Let's try to define them so that we can use these definitions as constraints for Multinomial Pascal's Triangle. I will try to arrange multinomial terms in $\mathbb{R}^n$.

Definition 1.1. *A* ***n-nomial Pascal's Triangle*** *is an arrangement of terms of* $(a_1 + a_2 + \cdots a_n)^k$ *for* $k \in \mathbb{N} \cup \{0\}$ *in some* $\mathbb{R}^m$*.*

We will use words multinomial and n-nomial interchangeably, with the use of n-nomial when n has to be chosen.

2 Defining Constraints

2.1 Explanation of visually appealing

The concept of "visually pleasing" is ambiguous at best. One can say that the core quality of the visually pleasing is that the arrangement of binomial terms forms an equilateral triangle. However, this is not open to generalization to higher dimensions. Here are some basic observations.

1. The terms are the same distance away from "adjacent" terms.
2. The location of the term follows a grid-like structure.

Now let's try to formulate these observations

Bound 1. *We see that $2ab$ in the above Pascal's Triangle is of the same distance to $1a, 1b, 1a^2, 1b^2, 3a^2b, 3ab^2$. These terms have the addition of a variable (ex: $ab \to a^2b$), deletion of a variable (ex: $ab \to b$) and substitution of variable (ex: $ab \to a^2$). We can define 'adjacent terms' as terms that are the one hamming distance away, and require that the distance to the adjacent terms to be same.*

Bound 2. *Let the position of a^0b^0 be the origin. Let's define $\vec{a}, \vec{b}$ to be the vector from the origin to the terms a, b. We see that a term with variables a^kb^l is located at $k\vec{a} + l\vec{b}$, creating a lattice.*

In particular, bound 2 is a strict bound: The entire arrangement is determined by the initial vectors.

Definition 2.1. *A Multinomial Pascal's Triangle satisfying bound 2 is called Lattice Multinomial Pascal's Triangle.*

2.2 Explanation of sum and "above"

A numerical characteristic of Pascal's Triangle is that the coefficient of the term is equal to the sum of the two coefficients of the terms above. Before we get further, let's define some terms.

2.2.1 Terms

In Pascal's Triangle, we see that the terms of $(a + b)^k$ form a line. Furthermore, the terms of $(a + b)^k$ can be said as located "above" terms of $(a + b)^{k+1}$.

Definition 2.2. *We denote the p-layer of n-nomial Pascal's Triangle as the location of terms of $(a_1 + a_2 + \cdots a_n)^k$*

From the observation above, we can create a bound where each layer exists in a hyperplane of $\mathbb{R}^n$, and that the aboveness can be explained by the location in the line orthogonal to the hyperplane.

Bound 3. *Let's say that n-noimal Pascal's Triangle is arranged in $\mathbb{R}^k$. The p-layer of n-nomial Pascal's Triangle lies on a hyperplane of $\mathbb{R}^k$.*

Remark 1. *Since 0-layer constitutes a single element, and thus can lie on any hyperplane, let's limit the hyperplane of the 0-layer to ones that are parallel to the 1-layer.*

As a final bound, we would want to arrange Pascal's Triangle in the lowest dimension possible.

Bound 4. *If an arrangement of n-nomial terms complying with the bounds is feasible in $\mathbb{R}^k$, then an arrangement that require $\mathbb{R}^p$ for $p > k$ is not considered a n-nomial Pascal's Triangle.*

2.2.2 Resolvation of Sums

In this subsection, we will show that if the arrangement of Multidimensional Pascal's Triangle satisfies the above three bounds, the property of the coefficient being the sum of the coefficients above is nicely satisfied and arranged.

Definition 2.3. *For a term k corresponding to $a_1^{x_i}, a_2^{x_2}, \cdots a_n^{x_n}$, the terms that are directly above k are terms that correspond to $\{a_1^{x_i}, a_2^{x_2}, \cdots a_n^{x_n}/a_i | 1 \le i \le n\}$*

Theorem 2.1.

$$\binom{n}{a_1, a_2, \ldots, a_k} = \sum_{i=1}^{k} \binom{n-1}{a_1, \ldots, a_i - 1, \ldots, a_k}$$

Proof. We can prove this by induction. However, there is a more intuitive proof for this. We see that $\binom{n}{a_1, a_2, \ldots, a_k}$ is the number of ways to divide n elements into k groups with a_i elements in the ith group. Let's choose the first element and divide the cases depending on which group it goes to. If it goes to the ith group, then we have $n - 1$ elements to divide into k groups with $a_i - 1$ elements in the ith group. This corresponds to the value of $\binom{n-1}{a_1, \ldots, a_i - 1, \ldots, a_k}$. □

Thus, the terms directly above term k have the sum of coefficients equal to the coefficient of k.

Remark 2. *For any term a, the terms that are directly above a are adjacent to a and is located in a layer above a.*

Thus with the constraints, we get an arrangement where the sum property of coefficients is satisfied, and the location of these summands are nicely located.

3 Existance of a Multinomial Pascal's Triangle

Let's now show that there exists an arrangement of multinomial terms that satisfy the above requirements.

First, let's look at the formulation of the zeroth and the first layer. Since $||\vec{a_i}||$ have to have same for all i, let's set $||\vec{a_i}|| = 1$.

Also, for any $i, j \in (1, 2, 3 \cdots n)$, $i \neq j$, the following condition must be satisfied by constraint 1.

$$||a_i|| = ||a_j|| = ||a_i - a_j||$$

With the second cosine law, we can see that

$$||a_i - a_j||^2 = ||a_i||^2 + ||a_j||^2 - 2||a_i||||a_j|| \cdot \cos(a_i a_j)$$
$$1 = 1 + 1 - 2 \cdot \frac{a_i \cdot a_j}{||a_i||||a_j||}$$
$$\frac{1}{2} = a_i \cdot a_j$$

Let's now show that such vectors a_n exist in R^n. Let's formulate the vectors so that a_j has values only at the first j items.

We set $a_1 = [1, 0, 0, \cdots 0]$. Then we see that a_2 has to have the first item as $\frac{1}{2}$ since $a_1 \cdot a_2 = \frac{1}{2}$. The second item is forced to be $\pm\frac{\sqrt{3}}{2}$. For our job, let's say that $a_2 = [\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \cdots 0]$.

Let's try one more. for a_3, the first two values are fixed from the dot product constraint. $a_3 = [\frac{1}{2} \cdot \frac{1}{\sqrt{12}}, \cdots]$. The third value is fixed from $||a_3|| = 1$.

With this method, creating a new a_i is always possible. These vectors satisfy the first constraint. Let's formally prove this by induction

Theorem 3.1. *Sequence of vectors that follow the following constraints*

$$\forall i, j \in [1, 2 \cdots n], a_i \cdot a_j = \frac{1}{2}$$
$$||a_k|| = 1$$

Is possible to formulate for any $n \in \mathbf{N}$.

Proof. Proof by induction. For the base case, we can set $a_1 = [1, \cdots, 0]$

Now let's say that $a_k = [a_k^1, a_k^2 \cdots a_k^k, 0, \cdots, 0]$ and show that there exists a_{k+1} that satisfy the bounds.

We can set $a_{k+1} = [a_k^1, a_k^2 \cdots a_k^{k-1}, \frac{1}{2} - \sum_{i=1}^{k-1}(a_k^i)^2, \pm\sqrt{1 - \sum_{i=1}^{k}(a_{k+1}^i)^2}, \cdots, 0]$.

We can prove that the dot product is always less than $\frac{1}{2}$ and that $||a_{k+1}|| < 1$ and that we can find a_{k+1}^k and a_{k+1}^{k+1}. □

Corollary 3.2. *In $\mathbf{R}^n$, if $a_1, a_2, \cdots a_{n-1}$ is defined, then there are only two choices for a_n.*

We have shown the existence of a set of vectors that satisfies the adjacent-same-distance constraint for layer 0 and 1. These vectors in fact, when used in grid-like formulation, the set of vectors can be used to define a n-nomial Pascal's Triangle in R^n.

When using a grid-like structure, the distance between two adjacent terms is of form

$$||\vec{a_i}||, ||\vec{a_i} - \vec{a_j}||$$

Which is satisfied by the constraint on the first two layers.

Before constraint 3, let's prove a simple observation

Lemma 3.3. *Any set of vectors a_i that satisfies the Mutinomial Pascal's Triangle must form a basis.*

Proof. For the sake of contradiction, let's say that a_i do not form a basis, that without loss of generality, there exists c_is, where

$$\sum_{i=1}^{n-1} c_i a_i = a_n$$

Then

$$a_n \cdot a_j = \frac{1}{2}$$
$$(\sum_{i=1}^{n-1} c_i a_i) \cdot a_j = \frac{1}{2}$$
$$\frac{1}{2}c_j + \sum_{i=1}^{n-1} c_i \frac{1}{2} = \frac{1}{2}$$
$$c_j + \sum_{i=1}^{n-1} c_i = 1$$

Also, we see that

$$a_n \cdot a_n = 1$$
$$a_n \cdot \sum_{i=1}^{n-1} c_i a_i = 1$$
$$\sum_{i=1}^{n-1} c_i \frac{1}{2} = 1$$
$$\sum_{i=1}^{n-1} c_i = 2$$

Thus we are left with an observation that $c_j = -1$ for some j in $i, \ldots, n-1$. This means that $\sum_{i=1}^{n-1} c_i = -(n-1) \neq 2$. Thus there is a contradiction. $\downarrow$ □

Now let's look at $k\vec{a_1}$. We see that the location of terms in the kth layer is in a linear combination of a_i where $i \in [2, 3, \cdots n]$. since a_k are linearly independent, layer lies in a unique hyperplane.

So there exists Multinomial Pascal's Triangle that satisfies the constraints

Theorem 3.4. *The n-nomial Pascal's Triangle must exist in $\mathbb{R}^n$.*

Proof. From Lemma3.3, the lower bound of dimension is n. Furthermore, there exists a n-nomial Pascal's Triangle in $\mathbb{R}^n$. Thus, by bound 4, all n-nomial Pascal's Triangle must lie in $\mathbb{R}^n$. □

So for now on bound 4 means a more strict condition, that n-nomial Pascal's Triangle lie in $\mathbb{R}^{\ltimes}$.

4 Generation of Lattice Multinomial Pascal's Triangle

In the last section, we showed the existence of the multinomial Pascal's Triangle that uses lattice structure. In this section, we show that the generation of all of these Pascal's Triangle is possible.

Lemma 4.1. *If set of vectors $\{v_1, v_2, \cdots, v_n\}$, $\{w_1, w_2 \cdots w_n\}$ form a basis in $\mathbf{R}^n$, then there exists a matrix W such that $W[v_1, v_2, \cdots, v_n] == [w_1, w_2, \cdots w_n]$*

Proof. Change of Basis Theorem in Linear Algebra □

Lemma 4.2. *Given a basis for Lattice Multinomial Pascal's Triangle, X and an orthogonal matrix A, AX creates another basis for Lattice Multinomial Pascal's Triangle*

Proof. Orthogonal matrix preserves relative angle and vector length. Thus, the new vectors AX also satisfies the bounds. □

Now, let us consider $V = \{v_1, v_2, \cdots, v_n\}$, $W = \{w_1, w_2 \cdots w_n\}$ where V, W are the vectors of Lattice Multinomial Pascal's Triangle. Then we see that any pair of vector q_1, q_2 and p_1, p_2 where $q_1 = a_1v_1 + a_2v_2 \cdots + a_nv_n$ and $p_1 = a_1w_1 + a_2w_2 + \cdots + a_nw_n$ and q_2, p_2 likewise must satisfy

$$q_1 \cdot q_2 = p_1 \cdot p_2$$

.

Furthermore, since V, W form basis in $\mathbb{R}^n$, there exists a change in basis matrix A. The change of basis matrix has to satisfy $q_1W = p_1, q_2W = p_2$. Thus

$$\begin{aligned} q_1 \cdot q_2 &= q_1A(q_2A)^T \\ q_1q_2^T &= q_1AA^Tq_2^T \end{aligned}$$

Thus A is an orthonormal matrix.

Lemma 4.3. *For any two basis of lattice multinomial pascal's triangle, A, B, the change of basis matrix W where $WA = B$ is an orthonormal matrix.*

Proof. Explanation above. □

Theorem 4.4. *An orbit of an orthonormal matrix with an element of basis of lattice n-nomial Pascal's Triangle generates the entire set of lattice n-nomial Pascal's Triangle.*

Proof. From Lemma 4.1, we see that there exists a change of basis matrix for every two elements in the basis of lattice n-nomial Pascal's Triangle. Furthermore, from Lemma 4.3, we showed that such matrix have to be orthonormal. Thus, an orbit of orthonormal matrix with basis of lattice n-nomial Pascal's Triangle generates the entire set. □

Theorem 4.5. *There exists a bijection between the basis of lattice n-nomial Pascal's Triangle and a set of orthonormal matrices.*

Proof. From 4.4, we see that the function from set of orthonormal matrices to the basis of lattice Pascal's Triangle is subjective.

The function is also injective as change of basis matrix is unique for each transformation. □

5 Loosening Bounds

5.1 Uniqueness of binomial Pascal's Triangle

We have shown all possible multinomial triangles that satisfy the three bounds. What happens if we loosen the lattice-structure bound?

Well, then we see that there are multiple formulations for a "possible" binomial Pascal's Triangle

$$\begin{matrix} & 1a & \\ 1a & & 1b \end{matrix}$$

$$\begin{matrix} 1a^2 & (1a, 2ab) & 1b^2 \\ 1a & & 1b \end{matrix}$$

$$\begin{matrix} & 1a^2 & (1a, 2ab) & 1b^2 & \\ 1a^3 & (1a, 3a^2b) & & (1b, 3ab^2) & 1b^3 \end{matrix}$$

We can flip every layer of Pascal's Triangle. However, this violates bound 3. In this section, we show that when the first layer is fixed, there exists a unique Pascal's Triangle following bounds 1,3,4

Definition 5.1. ***Initial Vectors*** *of n-nomial Pascal's Triangle is the vector from the origin to the first layer.*

From Bound 1, and section 4, we show that these vectors form a basis in $\mathbb{R}^n$.

Theorem 5.1. *There exists a unique binomial Pascal's Triangle given the initial vectors.*

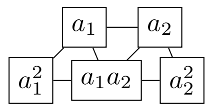
Proof. Layer 0 and layer 1 is fixed by the theorem proved above.

Let's look at layer 1 and layer 2. layer 2 has three terms and is of dimension 1. There are two possible configurations of layer 2. namely

$$a_1^2 - -a_1a_2 - -a_2^2$$
$$(a_1^2, a_2^2) - -a_1a_2$$

We will show later why the second configuration is not possible. Let's look at the first configuration to start.

Then adding layer 2 to layer 1, we are faced with this particular problem.



From the adjacentness constriant, every angle created by adjacent terms must be 60 degrees. Furthermore, $2a_1, a_1a_2, 2a_2$ form a line from constraint 3. We see that

$$(\overrightarrow{a_1^2, a_1a_2} - \overrightarrow{a_1^2, a_1})\overrightarrow{a_1, a_2^2} = \frac{1}{2}$$
$$\overrightarrow{a_1^2, a_1a_2} \cdot \overrightarrow{a_1, a_2^2} - \overrightarrow{a_1^2, a_1} \cdot \overrightarrow{a_1, a_2^2} = \frac{1}{2}$$
$$\overrightarrow{a_1^2, a_1a_2} \cdot \overrightarrow{a_1, a_2^2} = 1$$

Thus $\overrightarrow{a_1^2, a_1a_2}$ and $\overrightarrow{a_1, a_2^2}$ are parallel to each other.

First, notice that this implies that $\angle(a_1^2, a_1a_2, a_2) = 120$ degrees. iF a_1^2 and a_2^2 coincide in location, $|\overline{a_2a_2^2}| > 1$, showing that the second configuration is not applicable.

So we see that $\overrightarrow{a_1^2, a_1}$ and $\overrightarrow{a_1a_2}$ is 60 degrees. In $\mathbb{R}^2$, there are only two possible values for $\overrightarrow{a_1^2, a_1}$. Specifically, one value is $\overrightarrow{a_1, 0}$ and another is a reflection of the vector across $\overline{a_1a_2}$. A similar argument can be made of $\overrightarrow{a_2^2, a_2}$.

Since terms a_1^2, a_1a_2, a_2^2 must be in a line with distance 2, The vectors $\overrightarrow{a_1^2, a_1}, \overrightarrow{a_2^2, a_2}$ has to be both headed towards the origin or both not headed towards the origin. If both of them are not headed towards the origin, we have a violation of constraint 3. The origin coincides in a hyperplane formed by a_1^2, a_1a_2, a_2^2.

A similar argument can be made for adding other layers. In each layer, there is a unique way to attach the layer to the triangle.

Thus, there is only one way to formulate a binomial Pascal's Triangle. □

5.2 Uniqueness of n-nomial Pascal's Triangle

In this section, we show that in fact, even with the absence of bound 2, all n-nomial Pascal's Triangle is lattice n-nomial Pascal's Triangle. Let's first prove a lemma

Lemma 5.2. *We can create a bijective mapping of terms of layers up to k of (n-1) Pascal's Triangle. to terms of k layer of n-nomial Pascal's Triangle.*

Proof. The mapping sends $a_1^{j_1}a_2^{j_2}\cdots a_{n-1}^{j_{n-1}}$ to $a_1^{j_1}a_2^{j_2}\cdots a_{n-1}^{j_{n-1}}a_n^{k-\sum j_i}$.

The mapping is injective as the first $n-1$ variable's powers are conserved and the mapping is also surjective as for any term of k layer of n-nomial Pascal's Triangle, we can remove the last variable to get a reverse mapping. □

Lemma 5.3. *The adjacent distance constraint (Bound 1) is preserved in the bijective mapping of Lemma 5.2*

Proof. The adjacent terms can be categorized into addition, deletion, and substitution. For any item in the domain, say $a_1^{j_1} a_2^{j_2} \cdots a_{n-1}^{j_{n-1}}$, Addition corresponds to substitution of a_n with a_k for some $k \leq n-1$. Deletion corresponds to substitution of a_k with a_n. Substitution corresponds to a substitution of a_k, a_i for some $k, i \leq n-1$. □

Both the k-layer of n-nomial Pascal's Triangle and (n-1)-nomial Pascal's Triangle has to lie in $\mathbb{R}^{n-1}$. From the lemmas above, k-layer of n-nomial Pascal's Triangle has adjacentness constraint at least as harsh as (n-1)-nomial Pascal's Triangle. As k layer of n-nomial Pascal's Triangle has to satisfy adjacentness constraints from $k+1$ and $k-1$ layers, it has an additional constraint.

Let's now prove a simple lemma

Lemma 5.4. *There exists a unique kth layer of n nomial Pascal's Triangle, that satisfies $|\overline{a_i^k a_j^k}| = k$ for all i, j,*

Proof. First, such constraints can be met by following the formulation of Pascal's Triangle from section 3. Now for uniqueness, The formulation above forces the locations of any $a_i^p a_j^q$. This forces the location of $a_i^p a_j^q a_l^r$ which forces locations of terms with four variables. Continuing this argument till terms with n variables shows that location of the terms are fixed. □

This in fact, shows that if a layer satisfies $|\overline{a_i^k a_j^k}| = k$ for all i, j, that the layer follows same formation of lattice Pascal's Triangle.

We will show later, that all layers of Pascal's Triangle have to follow this form. For now, let's assume that all layers of Pascal's Triangle follow this form to show uniqueness of n-nomial Pascal's Triangle.

Theorem 5.5. *Initial vectors uniquely define n-nomial Pascal's Triangle, given that $|a_i^k a_j^k| = k$ for all $k \in \mathbb{N}$.*

Proof. We prove the following by induction.

We have already proven the uniqueness in 2-nomial Pascal's Triangle. Now let's assume that the formulation of Pascal's Triangle is unique to n-nomial and prove for (n+1)-nomial Pascal's Triangle. Notice that n-nomial Pascal's Triangle is lattice Pascal's Triangle.

We look at attaching kth layer of (n+1)-nomial Pascal's Triangle. Consider connecting a_i^{k-1} to a_i^k and a_j^{k-1} to a_j^k. Notice that $a_i^{k-1}, a_i^{k-2} a_j, \cdots a_j^{k-1}$ lies in line segment and $a_i^k, a_i^{k-1} a_j \cdots a_j^k$ also lies in a line segment.

Using a similar argument as 5.1, we can see that the two line segments are parallel to each other. Thus vector $\overrightarrow{a_i^k, a_i^{k-1}}$ must be 60 degrees from $\overrightarrow{a_i^{k-1}, a_i^{k-2} a_j}$.

We also notice that a_i^{k-1}, has vectors pointing to a_j^{k-1} for $j \neq i$ and $j \in \{1, 2, 3 \cdots n+1\}$. These vectors have to have a mutual dot product of $\frac{1}{2}$. the new vector $\overrightarrow{a_i^k, a_i^k}$ must also have the same condition as $\overrightarrow{a_i^k, a_i^{k-1}}$ must be 60 degrees from $\overrightarrow{a_i^{k-1}, a_i^{k-2} a_j}$.

According to 3.2, 5.2, there are only two possible vectors in $\mathbb{R}^{n+1}$ that satisfy this. namely a vector of $\overrightarrow{a_i^{k-1}, a_i^{k-2}}$ and vector that is a reflection upon the hyperplane. As shown in 5.1, Since $a_i^k, a_i^{k-1} a_j \cdots a_j^k$ is a line segment of length k, the other $\overrightarrow{a_j^k, a_j^{k-1}}$ must all either be $\overrightarrow{a_j^{k-1}, a_j^{k-2}}$ or all be a reflection onto the hyperplane. If every $\overrightarrow{a_j^k, a_j^{k-1}}$ is a reflection of $\overrightarrow{a_j^{k-1}, a_j^{k-2}}$ upon a hyperplane, then it coincides with the $k-2$ hyperplane and thus, there is only one way to add a layer to $(n+1)$-nomial Pascal's Triangle.

Furthermore, this addition satisfies the adjacency constraint between k and $k-1$ layers as the attachment is in fact, a lattice n-nomial Pascal's Triangle. □

We still need to prove that the only possible formulation for a layer of Pascal's Triangle is that of a lattice Pascal's Triangle. The proof is surprisingly similar to 5.5.

Theorem 5.6. *If the n-nomial Pascal's Triangle follows constraint of lattice Pascal's Triangle constraint until $k-1$th layer, then the kth layer must follow condition of 5.4.*

Proof. for the kth layer, fix the location of a_1^k and set it as the origin. Then allocation of the terms where power add up to k is similar to the allocation of terms of $n-1$ Pascal's Triangle.

The only difference is the lack of constraint 3 and the addition of constraints from the terms of $k-1$th layer. The two constraints resolve to have the same effect.

Let's define a l-sublayer as a subset of the k layer where terms include a_1^{k-l} (exclusive, does not include $a_1^{k-(l-1)}$ and more). The formulation problem can be simplified as the task of attaching $l+1$ sublayer to l.

Let's focus on attaching $a_1^{k-1}a_2$ to $a_1^{k-2}a_2^2$. Notice that $\overrightarrow{a_1^{k-2}a_2^2, a_1^{k-2}a_2a_j}$ must have mutual dot product of $\frac{1}{2}$ for all j.

With the same argument as 5.5, there are only two possible ways to attach $a_1^{k-1}a_2$ to $a_1^{k-2}a_2^2$. one is to attach it with the same vector attaching a_1^k to $a_1^{k-1}x_2$ and the other is to attach it with the vector reflected by hyperplane formed by the first layer.

The latter is not possible. $\overrightarrow{a_1^{k-1}, a_1^{k-2}a_2}$ has to be parallel to $\overrightarrow{a_1^k, a_1^{k-1}a_2}$ from the subproof of 5.5. The other choice results in $|\overrightarrow{a_1^{k-2}a_2, a_1^{k-2}a_2^2}| > 1$, violating bound 1.

Thus $\overrightarrow{a_1^k, a_1^{k-1}a_2} = \overrightarrow{a_1^{k-1}a_2, a_1^{k-2}a_2^2}$. Using the same argument, we show that the condition of 5.4 has to be met. □

Theorem 5.7. *Initial vectors uniquely define n-nomial Pascal's Triangle.*

Proof. From 5.6, we showed that given a lattice n-nomial Pascal's Triangle up to $k-1$th layer, the next layer has satisfy $|a_i^k a_j^k| = k$. 5.5 showed that attaching each layer is unique. Since every multinomial Pascal's Triangle up to layer 1 is a lattice multinomial Triangle, Every multinomial Pascal's Triangle is a lattice multinomial triangle. This implies that the initial vectors uniquely define the multinomial Pascal's Triangle. □